Equivariant definable triangulations of definable G sets

Tomohiro Kawakami
Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani Wakayama 640-8510, Japan

Received July 26,2005

Abstract

Let G be a finite group. We prove that every definable G set in a representation Ω of G admits an equivariant definable triangulation (L, ϕ) such that for each open simplex $int(\Delta)$ of L, $\phi(int(\Delta))$ is a locally closed definable C^r submanifold of Ω and that it induces a definable triangulation of X/G compatible with the orbit types.

2000 *Mathematics Subject Classification.* 14P10, 57S10, 57S15, 03C64.
Keywords and Phrases. Definable G sets, equivariant definable triangulations, o-minimal.

1 . Introduction

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [2], [4], see also [12]. Further properties and constructions of them are studied in [3], [5], [10]. It is known in [11] that there exist uncountably many o-minimal expansions of $\mathcal{R}$. Any definable category is a generalization of the semialgebraic category, the definable category on $\mathcal{R}$ coincides with the semialgebraic one, and this category is studied in [1]. An equivariant definable category is studied in [7], [8], [6].

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A *representation map* of a definable group G is a group homomorphism from G to some $O(n)$ which is definable. A *representation* of G is the representation space of a representation map of G. A *definable G set* means a G invariant definable subset of some representation of G.

In this paper, we are concerned with equivariant definable triangulations of definable G sets when G is a finite group.

Let X be a definable G set. A definable triangulation (L, ϕ) of the orbit space X/G is *compatible with the orbit types* if for any orbit type (H), $\phi \circ \pi(X(H))$ is a union of open simplexes of L, where $\pi : X \to X/G$ denotes the orbit map and $X(H) = \{x \in X | (G_x) = (H)\}$.

Let G be a finite group and X a definable G set. An *equivariant definable triangulation* (L, ϕ) of X consists of a G invariant union L of open simplexes of an equivariant simplicial complex and a definable G homeomorphism $\phi : |L| \to X$.

Theorem 1.1. *Let G be a finite group, X a definable G set in a representation Ω of G and r a positive integer. Then there exists an equivariant definable triangulation (L, ϕ) of X such that:*

(1) For any open simplex $int(\Delta^n)$ of L,

$\phi(int(\Delta^n))$ is a locally closed definable C^r submanifold of Ω and $\phi|int(\Delta^n)$ is a definable C^r diffeomorphism onto its image.

(2) *This triangulation induces a definable triangulation of X/G compatible with the orbit types.*

In particular, if X is compact, then we can take L to be an equivariant simplicial complex.

2. Proof of Theorem

Let G be a finite group. Recall the definition of equivariant simplicial complexes. A *simplicial G complex* consists of a simplicial complex K together with a G action $\psi : G \times K \to K$ such that $\psi_g = \psi(g, \cdot) : K \to K$ is a simplicial homeomorphism for any $g \in G$. We say that a simplicial G complex is an *equivariant simplicial complex* if the following two conditions are satisfied.

(1) For any subgroup H of G, if $\Delta^n =< v_0, \dots, v_n >$ and $\Delta^{n'} =< h_0v_0, \dots, h_nv_n >$ are simplexes of K for $h_i \in H$, then there exists an $h \in H$ such that $hv_i = h_iv_i$ for all i.

(2) For every simplex Δ^n of K, the vertices $v_0, \dots, v_n$ of Δ^n can be ordered with $G_{v_n} \subset \cdots \subset G_{v_0}$.

Note that the second barycentric subdivision of any simplicial G complex is an equivariant simplicial complex (e.g. [9])

To prove Theorem, we need results on the C^r singular point set of a definable set and equivariant stratifications of definable G sets.

Theorem 2.1. *(II.1.10 [12]) Let X be a nonempty definable set and $r \in \mathbb{N}$. Then the C^r singular point set is a definable subset of X whose dimension is less than $\dim X$, where $\dim \emptyset = -\infty$.*

Now we consider equivariant stratifications of definable G sets.

Theorem 2.2. *Let G be a compact definable group, X a definable G set in a representation Ω of G and $1 \le r < \infty$. Let $\{X_i\}_{i=1}^n$ be a finite collection of definable G subsets of X. Then there exists a finite partition $\{A_l\}_l$ of X into locally closed definable C^rG submanifolds of Ω such that:*

(1) Every X_i is a union of some A_l.

(2) If $A_j \cap A_k \neq \emptyset$, then $A_k \subset A_j$.

Proof. Refining $\{X_i\} \cup \{X\}$, if necessary, we may assume that $\{X_i\}$ is a partition of X.

We proceed by induction on $\dim X$. Then by Theorem 2.1, for each X_i, there exists a definable C^r submanifold N_i of contained in X_i and the dimension of $S_i := X_i - N_i$ is less than $\dim X_i$. Note that $N_i \supset S_i$. By the definition of C^r regular point sets, N_i is G invariant. Thus N_i is a locally closed definable C^rG submanifold of Ω. Hence S_i is a definable G set. Note that connected components of N_i do not necessarily have the same dimension. Let A_{it} be the union of connected components of N_i with dimension t. Then each A_{it} is a locally closed definable C^rG submanifold of Ω. Let $S_{i_j} = (A_{ij} - A_{ij}) \cap X_i$. Then S_{i_j} is a G invariant definable subset of X and $A_{kt} \cap S_{i_j} \neq \emptyset \Rightarrow S_{i_j} \subset A_{kt}$.

Let $Y = \cup S_{ij}$ and $Y_i = Y \cap X_i$. Since $\dim Y < \dim X$ and by the inductive hypothesis, there exists a finite partition $\{B_k\}$ of Y into locally closed definable C^rG submanifolds of Ω satisfying the conditions in the theorem. Therefore $\{A_l\} := \{A_{it}\} \cup \{B_k\}$ is the required partition of X.

Proof of Theorem 1.1. First we prove the case where X is compact. Note that for each orbit type (H) of X, $X(H) = \{x \in X | G_x$ is conjugate to $H\}$ is G invariant.
Thus by Theorem 2.2, there exists a finite partition $\{A_l\}$ of X into locally closed definable C^rG submanifolds of compatible with $\{X(H)\}$. Since A_l is a definable G set and the orbit map $\pi : X \to X/G$ is a definable map, $\pi(A_l)$ is a definable subset of X/G. Thus there exists a definable triangulation

(K, τ) of X/G compatible with $\{\pi(A_l)\}$ by 8.2.9 [2]. Replacing some subdivision, if necessary, for each $\Delta \in K, \pi^{-1}(\tau(\text{Int } \Delta))$ is a definable $C^r G$ submanifold of Ω and there exists a definable C^r section $s_\Delta : \tau(\Delta) \to X$ such that:

(1) $s_\Delta(\tau(\text{Int } \Delta))$ is a locally closed definable C^r submanifold of Ω.

(2) $s_\Delta | \pi(\text{Int } \Delta)$ is a definable C^r diffeomorphism onto its image.

Note that $s_\Delta(\tau(\Delta)) = (\pi^{-1}(\tau(\Delta)))^H$ and $s_\Delta(\tau(\text{Int } (\Delta))) = (\pi^{-1}(\tau(\text{Int } (\Delta))))^H$, where H is the orbit type of $\tau(\text{int } \Delta^n)$.

Let L' be a finite abstract simplicial complex whose simplexes are $\{gs(\tau(\Delta)) | \Delta \in K, g \in G\}$. Then L' is an abstract G complex with the following G action. For a simplex $\{hs(v_0), \dots, hs(v_n)\}$ in L' and $g \in G$, then $\phi_g(\{hs(v_0), \dots, hs(v_n)\}) = \{ghs(v_0), \dots ghs(v_n)\}$. Now let L be the realization of L' and let $<< gs(v_0), \dots, gs(v_n) >>$ denote the corresponding simplex to $\{gs(\tau(v_0)), \dots, gs(\tau(v_n))\}$ for $\Delta =< v_0, \dots, v_n > \in K$. Then L is an equivariant simplicial complex with the action induced by L'. For any $g \in G$, $\Delta^n =< v_0, \dots, v_n >$ in K and a section s on Δ^n, we define a linear homeomorphism $\prod_{<<gs(\Delta^n)>>} :<< gs(v_0), \dots, gs(v_n) >> \to \Delta^n, \prod_{<<gs(\Delta^n)>>}(gs(v_i)) = v_i$. Define $\prod = \cup \prod_{<<gs(\Delta^n)>>} : L \to K$. Then $\prod$ is a well-defined simplicial map. Note that $\prod : L \to K$ is the orbit map and $L/G = K$

Define a map $\eta : |L| \to X$ by $\eta| << gs(\Delta^n) >>= gs \circ \tau \circ \prod | << gs(\Delta^n) >>:<< gs(\Delta^n) >> \to gs((\tau(\Delta^n)), \Delta^n \in K, g \in G$. Then η is a well-defined definable G map. For each simplex $\Delta =<< gs(\Delta^n) >>$ of L, we have $\eta(\text{int } (\Delta)) = gs(\tau(\text{int } (\Delta))$. Moreover $\eta| << gs(\text{int } (\Delta^n))) >>$ is a definable $C^r G$ diffeomorphism from $<< gs(\text{int } (\Delta^n))) >>$ onto its image because it is the composition of a linear isomorphism $\prod | << gs(\text{int}\Delta^n) >>$, and definable C^r diffeomorphisms $\tau|\text{int } (\Delta)$ and gs. Therefore (L, η) is the required triangulation.

Assume that X is non-compact and closed in Ω. Then we may assume that $0 \notin X$. Let $\theta : \Omega - \{0\} \to \Omega - \{0\}$ be $\theta(x) = \frac{x}{||x||^2}$. Then it is a Nash G diffeomorphism because G acts on orthogonally, where $||x||$ denotes the standard norm of . Then $X^* = \theta(X) \cup \{0\}$ is a compact definable G set which is a one point compactification of X. Apply X^* to the compact case, we have an equivariant definable triangulation (K^*, ϕ^*). Then $(\phi^*)^{-1}(0)$ is a vertex. Since 0 is a fixed point, $(K^* - (\phi^*)^{-1}(0), \phi^* | K^* - (\phi^*)^{-1}(0))$ is the required triangulation.

Now we prove the case where X is not closed in Ω. Let X be the closure of X in Ω. Then X is a closed definable G set in Ω.

Applying X to the closed case, we have an equivariant triangulation of X compatible with X. Therefore we have the theorem.

References

[1] M. J. Choi, T. Kawakami and D.H. Park, *Equivariant semialgebraic vector bundles*, Topology Appl. **123** (2002), 383-400.

[2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

[3] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. Math. **140** (1994), 183–205.

[4] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.

[5] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.

[6] T. Kawakami, *Definable G CW complex structures of definable G sets and their applications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **55**, (2004), 1–15.

[7] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.

[8] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183–201.

[9] K. Kawakubo, *The theory of transformation groups*, Oxford Univ. Press, 1991.

[10] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), 4397–4419.

[11] J.P. Rolin, P. Speissegger and A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751-777.

[12] M. Shiota, *Geometry of subanalyitc and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.